

On the von Neumann Inequality for Linear Matrix Functions of Several Variables*

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Abstract. The theorem on the existence of three commuting contractions on a Hilbert space and of a linear homogeneous matrix function of three independent variables for which the generalized von Neumann inequality fails is proved.

J. von Neumann has shown in [1] that for any contractive linear operator T on a Hilbert space (i. e., T such that $\|T\| \leq 1$) and for any polynomial $p(z)$ of a complex variable the following inequality holds:

$$\|p(T)\| \leq \max_{z \in \Delta} |p(z)|,$$

where $\Delta = \{z \in \mathbb{C} : |z| \leq 1\}$ denotes the closed unit disk.

T. Ando [2] generalized this inequality to the case of any two commuting contractions T_1 and T_2 on a Hilbert space and any polynomial $p(z_1, z_2)$ of two independent variables:

$$\|p(T_1, T_2)\| \leq \max_{z \in \Delta^2} |p(z)|,$$

(here and further for any positive integer N $\Delta^N = \{z \in \mathbb{C}^N : |z_k| \leq 1, k = 1, \dots, N\}$ denotes the closed unit polydisk). N. Varopoulos [3], however, has shown that already for three commuting contractions an analogous inequality, in general, fails, namely, he has constructed the triple of commuting contractive linear operators T_1, T_2, T_3 on some finite-dimensional Hilbert space and the homogeneous polynomial of second degree $p(z_1, z_2, z_3)$ of three independent variables, such that

$$\|p(T_1, T_2, T_3)\| > \max_{z \in \Delta^3} |p(z)|.$$

Remark 1. The degree of a polynomial in Varopoulos's example can not be diminished. Indeed, let $\mathbf{T} = \{T_1, \dots, T_N\}$ be an N -tuple of contractions on some Hilbert space \mathcal{H} with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and let $l(z_1, \dots, z_N) = a_0 + \sum_{k=1}^N a_k z_k$ be an arbitrary polynomial of first degree of N independent variables, i. e., a linear scalar function on \mathbb{C}^N . Then

$$\|l(T_1, \dots, T_N)\| = \|a_0 I_{\mathcal{H}} + \sum_{k=1}^N a_k T_k\| = \sup_{\|x\|=\|y\|=1} |\langle (a_0 I_{\mathcal{H}} + \sum_{k=1}^N a_k T_k) x, y \rangle_{\mathcal{H}}|$$

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$$\begin{aligned}
&= \sup_{\|x\|=\|y\|=1} |a_0 \langle x, y \rangle_{\mathcal{H}} + \sum_{k=1}^N a_k \langle T_k x, y \rangle_{\mathcal{H}}| \leq \max_{\lambda \in \Delta, z \in \Delta^N} |a_0 \lambda + \sum_{k=1}^N a_k z_k| \\
&= \max_{|\zeta|=1, z \in \Delta^N} |a_0 \zeta + \sum_{k=1}^N a_k z_k| = \max_{z \in \Delta^N} |a_0 + \sum_{k=1}^N a_k z_k| = \max_{z \in \Delta^N} |l(z)|
\end{aligned}$$

(here we used the Cauchy–Bunyakovskiy–Schwartz inequality for the estimate of an inner product and the maximum modulus principle for analytic functions in the disk). It should be noted that the commutativity of an N –tuple \mathbf{T} is unessential in this calculation.

However, one may consider matrix–valued polynomials (i. e., polynomials with matrix coefficients) of several independent variables and the notion of such polynomial of several commuting contractions [4], and then, as we will show, the degree of a polynomial for which the corresponding generalized von Neumann inequality fails can be diminished to one. The statement of this problem has arisen under the consideration of multiparameter passive scattering linear systems (the definition of such systems one may find in [5]), however, it seems that it may have an independent interest, too.

Thus, let $\mathbf{T} = \{T_1, \dots, T_N\}$ be an N –tuple of commuting contractions on a Hilbert space \mathcal{H} , $P(z_1, \dots, z_N)$ be a matrix–valued polynomial of N independent variables, i. e.,

$$P(z) = \sum_{t \in \mathbb{Z}_+^N, |t| \leq m} A_t z^t \quad (z = (z_1, \dots, z_N) \in \mathbb{C}^N), \quad (1)$$

where $\mathbb{Z}_+^N = \{t \in \mathbb{Z}^N : t_k \geq 0, k = 1, \dots, N\}$; for $t \in \mathbb{Z}_+^N$ $|t| = \sum_{k=1}^N t_k$, $z^t = \prod_{k=1}^N z_k^{t_k}$, $A_t \in M_n(\mathbb{C}) = \mathcal{L}(\mathbb{C}^n)$, so that we shall identify the algebra of $n \times n$ matrices over \mathbb{C} with the C^* –algebra of all linear operators on the finite–dimensional Hilbert space \mathbb{C}^n . Define an operator

$$P(\mathbf{T}) = P(T_1, \dots, T_N) = \sum_{t \in \mathbb{Z}_+^N, |t| \leq m} A_t \otimes \mathbf{T}^t \in \mathcal{L}(\mathbb{C}^n \otimes \mathcal{H}) \cong \mathcal{L}(\mathcal{H}^n) \quad (2)$$

(here $\mathbf{T}^t = \prod_{k=1}^N T_k^{t_k}$). One may consider $P(\mathbf{T})$ as an element of the C^* –algebra $M_n(\mathbb{C}) \otimes \mathcal{L}(\mathcal{H}) \cong M_n(\mathcal{L}(\mathcal{H}))$ of all $n \times n$ matrices over $\mathcal{L}(\mathcal{H})$, i. e., as a matrix with operator entries (see [4]). We shall also use the following

Definition 1 [6]. An N –tuple $\mathbf{U} = \{U_1, \dots, U_N\}$ of commuting unitary operators on a Hilbert space \mathcal{K} is called a *unitary dilation* of an N –tuple $\mathbf{T} = \{T_1, \dots, T_N\}$ of commuting contractive operators on a Hilbert space $\mathcal{H} \subset \mathcal{K}$ if

$$\forall t \in \mathbb{Z}_+^N \quad \mathbf{T}^t = P_{\mathcal{H}} \mathbf{U}^t | \mathcal{H},$$

where $P_{\mathcal{H}}$ is an orthogonal projection onto \mathcal{H} in \mathcal{K} .

Remark 2. It follows from the general dilation theorem of W. Arveson (see [7], p. 278) that an N –tuple $\mathbf{T} = \{T_1, \dots, T_N\}$ of commuting contractions on a Hilbert space allows

a unitary dilation if and only if for any matrix-valued polynomial $P(z)$ of the form (1) the following generalized von Neumann inequality holds:

$$\|P(\mathbf{T})\| \leq \max_{z \in \Delta^N} \|P(z)\|, \quad (3)$$

where $P(\mathbf{T})$ is defined by equality (2). In particular, for $N = 1$ and $N = 2$ it means (see [8] and [2]) that inequality (3) always holds.

For $N = 3$ we shall formulate now the main statement of the present paper.

Theorem. *There exist a triple $\mathbf{T} = \{T_1, T_2, T_3\}$ of commuting contractions on some finite-dimensional Hilbert space \mathcal{H} and a triple $\mathbf{A} = \{A_1, A_2, A_3\}$ of $n \times n$ matrices over \mathbb{C} with some integer $n > 1$ such that a linear homogeneous matrix function $L(z_1, z_2, z_3) = A_1 z_1 + A_2 z_2 + A_3 z_3$ satisfies the inequality*

$$\|L(\mathbf{T})\| > \max_{z \in \Delta^3} \|L(z)\|. \quad (4)$$

For the proof we shall use the following definitions.

Definition 2 (e. g., see [9]). Let \mathcal{A} and \mathcal{B} be C^* -algebras. A linear map $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is called *positive* if it transforms positive elements of \mathcal{A} to positive elements of \mathcal{B} , i. e., $a \in \mathcal{A}$ and $a \geq 0$ imply $\varphi(a) \geq 0$.

Let \mathcal{S} be a linear subspace in a C^* -algebra \mathcal{A} (possibly, coinciding with \mathcal{A}), \mathcal{B} be a C^* -algebra, $\varphi : \mathcal{S} \rightarrow \mathcal{B}$ be a linear map. Then for every positive integer n $M_n(\mathbb{C}) \otimes \mathcal{A} \cong M_n(\mathcal{A})$ is the C^* -algebra of all $n \times n$ matrices over \mathcal{A} and $M_n \otimes \mathcal{S}$ is a linear subspace of this C^* -algebra. If id_n denotes the identity map of $M_n(\mathbb{C})$ onto itself, then $\varphi_n = id_n \otimes \varphi$ is a linear map of $M_n(\mathbb{C}) \otimes \mathcal{S}$ into $M_n(\mathbb{C}) \otimes \mathcal{B}$ which transforms matrices over \mathcal{S} to matrices over \mathcal{B} applying a linear map φ element by element.

Definition 3 [9]. A linear map $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is called *completely positive* if maps φ_n are positive for all integer $n \geq 1$.

Definition 4 [4]. A linear map $\varphi : \mathcal{S} \rightarrow \mathcal{B}$ is called *completely contractive* if maps φ_n are contractive (i. e., $\|\varphi_n\| \leq 1$) for all integer $n \geq 1$.

Proof of the Theorem. It follows from Remark 2 that a desired triple of contractions \mathbf{T} doesn't allow a unitary dilation. Let us take such triple of contractions from the known S. Parrott's example [10]. Let \mathfrak{X} be an arbitrary Hilbert space of dimension more than one (\mathfrak{X} may as well be taken finite-dimensional). We shall define $\mathcal{H} = \mathfrak{X} \oplus \mathfrak{X}$ and a triple $\mathbf{T} = \{T_1, T_2, T_3\}$ of contractive linear operators on a Hilbert space \mathcal{H} as block matrices:

$$T_1 = \begin{pmatrix} 0 & 0 \\ B_1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 \\ B_2 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 \\ I_{\mathfrak{X}} & 0 \end{pmatrix},$$

where B_1 and B_2 are arbitrary non-commuting unitary operators on \mathfrak{X} , $I_{\mathfrak{X}}$ is the identity operator on \mathfrak{X} . Evidently, T_k are commuting operators because $T_k T_j = 0$ ($k, j = 1, 2, 3$).

1, 2, 3). Let $\mathcal{S} = \text{span}\{\Lambda_1, \Lambda_2, \mathbb{I}\}$ be a three-dimensional linear subspace of the C^* -algebra $C(\mathbb{T}^2)$ of all continuous complex-valued functions on the two-dimensional torus \mathbb{T}^2 ($\mathbb{T}^N = \{\lambda \in \mathbb{C}^N : |\lambda_k| = 1, k = 1, \dots, N\}$ denotes the N -dimensional torus) generated by the functions $\Lambda_1(\lambda) = \lambda_1$, $\Lambda_2(\lambda) = \lambda_2$, $\mathbb{I}(\lambda) = 1$. We shall define the linear map $\varphi : \mathcal{S} \rightarrow \mathcal{L}(\mathfrak{X})$ by the following way: for any $\alpha, \beta, \gamma \in \mathbb{C}$ let

$$\varphi(\alpha \Lambda_1 + \beta \Lambda_2 + \gamma \mathbb{I}) = \alpha B_1 + \beta B_2 + \gamma I_{\mathfrak{X}}.$$

We shall show that the map φ is contractive but it is not completely contractive. According to Remark 1,

$$\begin{aligned} \|\varphi(\alpha \Lambda_1 + \beta \Lambda_2 + \gamma \mathbb{I})\| &= \|\alpha B_1 + \beta B_2 + \gamma I_{\mathfrak{X}}\| \leq \max_{z \in \Delta^2} |\alpha z_1 + \beta z_2 + \gamma| \\ &= \max_{\lambda \in \mathbb{T}^2} |\alpha \lambda_1 + \beta \lambda_2 + \gamma| = \|\alpha \Lambda_1 + \beta \Lambda_2 + \gamma \mathbb{I}\| \end{aligned}$$

(we used also the maximum modulus principle for analytic functions on the bidisk). Thus, φ is a contractive map. Let us suppose that φ is a completely contractive map also. Then, by Theorem 1.2.9 [4], it can be extended to a completely positive map $\tilde{\varphi} : C(\mathbb{T}^2) \rightarrow \mathcal{L}(\mathfrak{X})$, where $\tilde{\varphi}(\mathbb{I}) = I_{\mathfrak{X}}$. But such map has the form [9]:

$$\forall f \in C(\mathbb{T}^2) \quad \tilde{\varphi}(f) = P_{\mathfrak{X}} \pi(f) | \mathfrak{X},$$

where π is a representation of the C^* -algebra $C(\mathbb{T}^2)$ on some Hilbert space $\mathcal{Y} \supset \mathfrak{X}$, and $P_{\mathfrak{X}}$ is an orthogonal projection onto \mathfrak{X} in \mathcal{Y} . For any $x \in \mathfrak{X}$ we have:

$$\|x\| = \|B_k x\| = \|\varphi(\Lambda_k) x\| = \|\tilde{\varphi}(\Lambda_k) x\| = \|P_{\mathfrak{X}} \pi(\Lambda_k) x\| \quad (k = 1, 2).$$

On the other hand, since Λ_k are unitary elements of the C^* -algebra $C(\mathbb{T}^2)$, $\pi(\Lambda_k)$ are unitary operators on \mathcal{Y} . Therefore $\|x\| = \|\pi(\Lambda_k) x\|$. Thus $\|P_{\mathfrak{X}} \pi(\Lambda_k) x\| = \|\pi(\Lambda_k) x\|$, but this is possible only if $P_{\mathfrak{X}} \pi(\Lambda_k) x = \pi(\Lambda_k) x$, i. e., $\pi(\Lambda_k) x \in \mathfrak{X}$. Since x is an arbitrary element of \mathfrak{X} , from this we obtain that

$$\pi(\Lambda_k) | \mathfrak{X} = P_{\mathfrak{X}} \pi(\Lambda_k) | \mathfrak{X} = \tilde{\varphi}(\Lambda_k) = \varphi(\Lambda_k) = B_k \quad (k = 1, 2).$$

Hence we have

$$B_1 B_2 = \pi(\Lambda_1) \pi(\Lambda_2) | \mathfrak{X} = \pi(\Lambda_1 \Lambda_2) | \mathfrak{X} = \pi(\Lambda_2 \Lambda_1) | \mathfrak{X} = \pi(\Lambda_2) \pi(\Lambda_1) | \mathfrak{X} = B_2 B_1,$$

but this contradicts to the choice of operators B_1 and B_2 . Therefore φ is not a completely contractive map, i. e., there is an integer $n > 1$ for which $\|\varphi_n\| > 1$. This, in turn, means that there is a triple $\mathbf{A} = \{A_1, A_2, A_3\}$ of $n \times n$ matrices over \mathbb{C} such that

$$\|\varphi_n(A_1 \otimes \Lambda_1 + A_2 \otimes \Lambda_2 + A_3 \otimes \mathbb{I})\| > \|A_1 \otimes \Lambda_1 + A_2 \otimes \Lambda_2 + A_3 \otimes \mathbb{I}\|. \quad (5)$$

We shall investigate separately the left-hand side and the right-hand side of this inequality.

$$\begin{aligned} \|\varphi_n(A_1 \otimes \Lambda_1 + A_2 \otimes \Lambda_2 + A_3 \otimes \mathbb{I})\| &= \|A_1 \otimes B_1 + A_2 \otimes B_2 + A_3 \otimes I_{\mathfrak{X}}\| \\ &= \|(I_n \otimes P_{\{0\} \oplus \mathfrak{X}})(A_1 \otimes T_1 + A_2 \otimes T_2 + A_3 \otimes T_3) | \mathbb{C}^n \otimes (\mathfrak{X} \oplus \{0\})\| \\ &\leq \|A_1 \otimes T_1 + A_2 \otimes T_2 + A_3 \otimes T_3\|, \end{aligned}$$

where I_n is the identity $n \times n$ matrix, $\mathfrak{X} \oplus \{0\}$ and $\{0\} \oplus \mathfrak{X}$ are subspaces in $\mathcal{H} = \mathfrak{X} \oplus \mathfrak{X}$ of all such vectors $h = x_1 \oplus x_2$ that $x_2 = 0$ (resp. $x_1 = 0$), and $P_{\{0\} \oplus \mathfrak{X}}$ is the orthogonal projection in \mathcal{H} onto the second one of these subspaces. The right-hand side of (5)

$$\|A_1 \otimes \Lambda_1 + A_2 \otimes \Lambda_2 + A_3 \otimes \mathbb{I}\| = \max_{\lambda \in \mathbb{T}^2} \|A_1 \lambda_1 + A_2 \lambda_2 + A_3\|,$$

by virtue of the isomorphism between the C^* -algebra $M_n(\mathbb{C}) \otimes C(\mathbb{T}^2)$ and the C^* -algebra $C(\mathbb{T}^2, M_n(\mathbb{C}))$ of all continuous $n \times n$ matrix functions on \mathbb{T}^2 (see, e. g., proposition 4.7.3 [11]). But

$$\begin{aligned} \max_{\lambda \in \mathbb{T}^2} \|A_1 \lambda_1 + A_2 \lambda_2 + A_3\| &= \max_{\zeta \in \mathbb{T}^3} \|A_1 \zeta_1 + A_2 \zeta_2 + A_3 \zeta_3\| \\ &= \max_{z \in \Delta^3} \|A_1 z_1 + A_2 z_2 + A_3 z_3\|, \end{aligned}$$

according to the maximum norm principle for analytic matrix functions of several variables (see, e. g., [12]).

Thus, finally we obtain the inequality

$$\|A_1 \otimes T_1 + A_2 \otimes T_2 + A_3 \otimes T_3\| > \max_{z \in \Delta^3} \|A_1 z_1 + A_2 z_2 + A_3 z_3\|,$$

which coincides with (4) for the linear matrix-function $L(z_1, z_2, z_3) = A_1 z_1 + A_2 z_2 + A_3 z_3$. The proof of the theorem is complete.

References

- [1] von Neumann, J., *Eine Spectraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr., 4 (1951), 258–281.
- [2] Ando, T., *On a pair of commutative contractions*, Ann. of Math. 24 (1963), 88–90.
- [3] Varopoulos, N.Th., *On an inequality of von Neumann and an application of the metric theory of tensor product to operator theory*, Jour. Func. Anal. 16 (1974), 83–100.
- [4] Arveson, W.B., *Subalgebras of C^* -algebras*, Acta Math., 123 (1969), 141–224.
- [5] Kalyuzhniy, D., *Multiparameter passive scattering linear stationary dynamic systems (discrete case)*, Mark Krein International Conference "Operator theory and applications", Odessa, August 18–22, 1997. Book of abstracts.
- [6] Sz.-Nagy, B. and Foias, C., *Harmonic Analysis of Operators on Hilbert Spaces*, North-Holland, Amsterdam–London, 1970.
- [7] Arveson, W. B., *Subalgebras of C^* -algebras II*, Acta Math. 128 (1972), 271–308.
- [8] Sz.-Nagy, B., *Sur les contractions de l'espace de Hilbert*, Acta Sci. Math. 15 (1953), 87–92.

- [9] Stinespring, W., *Positive functions on C^* -algebras*, Proc. Amer. Math. Soc. 6 (1955), 211–216.
- [10] Parrott, S., *Unitary dilations for commuting contractions*, Pacif. J. Math. 34 (1970), 481–490.
- [11] Takesaki, M., *Theory of Operator Algebras*, v.I, Springer–Verlag, New-York–Heidelberg–Berlin, 1979.
- [12] Schwartz, L., *Analyse Mathématique*, v.II, Hermann, Paris, 1967.

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